Wheel Counting to Determine Pendulum Length

By Lloyd L. Lehn  PhD, CC

Author's Comment: This article was originally published in Horological Times. However, due to an administrative glitch, all of the Greek Pi's showed up as bold faced “P’s”. A correction note was published in the next issue and the editor offered to send a good copy to anyone who wanted it. The version printed here includes the Greek Pi's. It’s published here so others may benefit from reading it.

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Purpose:

The purpose of this article is to attempt to take some of the “black art” out of wheel counting.

I have been studying and repairing clocks for the past several years. During the period when I was studying to take the Certified Clockmakers exam, I become frustrated with the material I was able to locate on wheel counting. In general, the literature:

1) Assumed that I understood much more than I do - it’s not always obvious to me that some wheel is an idler and can be ignored;

2) Assumed it was OK to drop terms from equations because it was “obvious” - not so to me;

3) Almost totally ignored the units of the variables involved - the author just performed arithmetic on numbers with little or no discussion about the units involved.

My engineering education taught me that this is a very dangerous thing to do; and

4) Occasionally used “magic numbers” with no reference to the units of those magic numbers nor how they were derived.

Wheel counting articles are generally focused on calculating the length of a missing pendulum. An alternative to counting is to use trial and error - but that could take a lot of time. A quicker way is to sketch out the train from the center wheel to the escapement, count all the teeth involved and then make a few calculations to come up with a theoretical pendulum length. This at least gets you into the ball park from which you can proceed with trial and error to find a pendulum which will provide the correct time.

I have been frustrated by most of these articles so I decided to write my own. What better way to prepare for an exam than explaining the material to someone else?
**Approach:**

My approach will be to:

1) Create a complex gear train that will illustrate the concepts of my approach;

2) Look at the basic equations of motion of a pendulum and establish some basic constants - while always carrying along the units of the value involved;

3) Analyze the gear train motion using a technique where tables are used to represent equations. This provides some discipline to carrying along the units of the variables;

4) Calculate a table of constants for converting [beats/unit of time] into pendulum length for a number of situations; and

5) Calculate the theoretical length of the pendulum for the illustrative gear train.

**The Gear Train**

I’ve made up a gear train to illustrate the method I am going to suggest for calculating a pendulum length (see figure and table). My intent was to create a train more complex than those found in the real world. Hopefully, if my technique works for this train it should work for any train.

**Table 1 - Gear Train Inventory**

<table>
<thead>
<tr>
<th>Arbor Name</th>
<th>Arbor Code</th>
<th>Teeth Gear</th>
<th>Leaves Pinion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center</td>
<td>A</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>Idler</td>
<td>B</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2nd</td>
<td>C</td>
<td>78</td>
<td>8</td>
</tr>
<tr>
<td>3rd</td>
<td>D</td>
<td>72</td>
<td>7</td>
</tr>
<tr>
<td>Idler</td>
<td>E</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>Escape</td>
<td>F</td>
<td>30</td>
<td>6</td>
</tr>
<tr>
<td>Main</td>
<td>80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What do we know about this train?

- It has seven arbors and 10 gears/pinions.

- Three arbors - C, D, & F - have both a gear and a pinion.

- The 80 teeth on the main gear power the entire train through the pinion on Arbor C.

- The train would run down quickly were it not for the pendulum and pallets restraining the escape wheel on Arbor F. The pallets control the run down rate.

- When the pendulum is the correct length, Arbor A and the minute hand will make
Overall Approach

Let’s look at the entire gear train from Arbor A to Arbor F.

At A we have the revolutions of Arbor A per hour [Ra/hour]. At F we have beats per revolution of Arbor F [beats/Rf]. This is encouraging. We have both ends of the system [beats/Rf] and [Ra/hour] expressed in terms which might cancel out if we analyzed one arbor after another.

We know the relationship of the relative motion between adjacent arbors because we know how many teeth are on each gear/pinion. It seems reasonable that we should be able to work this out so that we end up with [beats/hour] by cancelling out all the revolutions of the arbors. Then we need to come up with some scheme to convert the [beats/hour] to the length of a pendulum.

We will do just that.

Equations and Constants

First let’s look at the basic equations involved.

The period of a theoretical pendulum is given by the basic physics equation:

\[
t_p = 2\pi \sqrt{\frac{L}{g}}
\]

Where:

- \(t_p\) - period of a pendulum [sec/period] - this is the complete period or cycle. In a pendulum clock, this is the time it takes for both a tick and a tock, i.e. two beats.
- \(\pi\) - 3.1416 - constant
- \(L\) - pendulum length [feet] and
- \(g\) - 32.2 ft/sec\(^2\) - constant (varies by location)

Expressed in units of the variables, Eq. 1 becomes:

\[
[\text{sec/period}] = [\text{?}] \left(\frac{\text{ft}}{\text{ft/sec}^2}\right)^{1/2}
\]

Note: I have used square brackets, [], for units and will do this throughout this article.

You should also note that I have put a question mark for the units of \(\pi\). Normally, \(\pi\) is considered a dimensionless constant. However, if one looks into the derivation Eq. 1, the term which includes the \(\pi\) provides the “per period” units. We will carry them along with the \(\pi\).

Thus Eq. 2 becomes

\[
[\text{sec/period}] = [1/\text{period}] \left(\frac{\text{ft}}{\text{ft/sec}^2}\right)^{1/2}
\]
But we really aren’t interested in calculating \( t_p \). This is easy to fix since we have two beats per period of the pendulum or:

Eq. 4
\[
[\text{sec/beat}] = [\text{sec/period}] \left( \frac{1 \text{ period}}{2 \text{ beat}} \right)
\]

or

Eq. 5
\[
t = \frac{t_p}{2} = \sqrt{\frac{L}{g}}
\]

where \( t \) is now [sec/beat] or

\[
[\text{sec/beat}] = \left( \frac{1}{\text{beat}} \right) \left( \frac{[\text{ft}]}{[\text{ft/sec}^2]} \right) ^{1/2}
\]

Note: We carry along the units of \( \pi \) as [1/beat]

We still haven’t reached the equation we’d like since we’re interested in \( L \).

Some simple algebraic manipulation results in:

Eq. 6
\[
L = \frac{t^2 g}{\pi^2} = t^2 \left( \frac{g}{\pi^2} \right)
\]

which we can express in units as:

Eq. 7
\[
[\text{ft}] = [\text{sec/beat}]^2 \left( \frac{[\text{ft}]}{[\text{sec}^2]} \right) \left( \frac{[\text{beat}]}{1} \right)^2
\]

or

Eq. 8
\[
[\text{ft}] = [\text{sec}^2/\text{beat}] \left( \frac{[\text{ft}]}{[\text{sec}^2]} \right) \left( \frac{[\text{beat}]}{1} \right) = [\text{ft}]
\]

Eqs. 6-8 are very nice equations to have for the elements of \( (g/\pi^2) \) are both well known constants. The combination should be easy to calculate - provided we maintain an awareness of the units for \( g \) and \( \pi \). The first term “\( t^2 \)” is expressed in [sec/beat]^2 or of we invert it \( (1/[\text{beats/sec}]^2) \).

[beats/sec]!

[beats/sec] looks like something we should be able to get from the gear train.

Note: I have been using feet for a linear measurement. We could have used inches, mm, meters or any linear measurement. Similarly, I have been working in seconds. Minutes or hours would work just as well. I used feet and seconds simply because most people recognize the value of \( g \) as 32.2 [feet/sec^2]. However, the value of \( g \) varies from one geographic location to another. I will use 32.2. The reader can adapt to local conditions.

**Gear Train Analysis**

Let us now look at the relationship between the relative motion of the arbors in our gear train.

Let us assume that we have two gears X and Y working together with the following parameters:

<table>
<thead>
<tr>
<th>Gear</th>
<th>Teeth</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>12</td>
</tr>
<tr>
<td>Y</td>
<td>48</td>
</tr>
</tbody>
</table>

If we focus at the point where they intersect, we see that for each tooth of X passing that point, exactly one tooth of Y also passes that point. When 12 teeth of X have passed the point, 12 teeth of Y will have passed the point. X will have made one complete revolution and Y will have made \( (12/48) \) or \( (1/4) \) of a revolution. This can be written as:

Eq. 9
\[
\frac{R_x}{R_y} = \frac{1}{(12/48)} = \frac{48}{12} = \frac{t_y}{t_x}
\]

where

\( R_x \) - revolutions of Arbor X

\( R_y \) - revolutions of Arbor Y

\( t_x \) - number of teeth on Gear X

\( t_y \) - number of teeth on Gear Y

We can say that the relative speed or rotation between two gears is inversely proportional to the number of teeth on each gear.
Tabular Notation

This is a good point to introduce the concept of using tabular notation for the gear equations. It’s a good idea to use this technique so that when we analyze a gear train we do not forget any important factor in the train analysis - nor do we mix up the units of the variables involved.

We could have written this in tabular notation as:

<table>
<thead>
<tr>
<th>Num</th>
<th>Rx</th>
<th>1</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denom</td>
<td>Ry</td>
<td>12/48</td>
<td>12</td>
</tr>
<tr>
<td>Num</td>
<td>rev x</td>
<td>t_y</td>
<td></td>
</tr>
<tr>
<td>Denom</td>
<td>rev y</td>
<td>t_x / t_y</td>
<td>t_x</td>
</tr>
</tbody>
</table>

The first two rows represent the numerator and the denominator of Eq. 9. The next two rows represent the same equation. However, the cells contain the units of the values in the first two rows. The narrow vertical lines represent the equal signs.

“So what?” you say.

As we progress through the analysis of the gear trains, I think you will see that this will be very useful. It will discipline us to be sure we express terms in the proper units - especially before we start cancelling out terms and values.

Let’s return to our gear train.

We want to work from one end to the other.

We could start from Arbor A or the Arbor F. However, I know that I am looking for something expressed in [beats/time unit]. By starting with F, I can put the unit of beats in the numerator which is where I would like to end up.

Let’s look at the escapement and Arbor F.

There are 30 teeth on F and two pallets on the anchor. Thus for each beat of the anchor we have the tabular equation:

<table>
<thead>
<tr>
<th>Num</th>
<th>2 x 30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denom</td>
<td>1</td>
</tr>
<tr>
<td>Units Num</td>
<td>Beat</td>
</tr>
<tr>
<td>Units Denom</td>
<td>Rf</td>
</tr>
</tbody>
</table>

In plain English, the last column points out that there are 60 beats per revolution of Arbor F. That’s neat.
Now let's move on to Arbor E. We will carry along what we know about Arbor F in the tabular equation.

The relative motion between arbors E and F are governed by the 6 leaf pinion and and the 23 tooth gear. The tabular equation expands to:

<table>
<thead>
<tr>
<th>Num</th>
<th>Denom</th>
<th>Units Num</th>
<th>Units Denom</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 x 30</td>
<td>1</td>
<td>Beat</td>
<td>Rf</td>
</tr>
</tbody>
</table>

Looking at the right column we see that F rotates \((23/6)\) time the number of revolutions of D. That makes sense since the pinion on F has fewer teeth and will be running faster than E.

At this point, we will drop the two left hand columns strictly for space considerations. If you are doing this manually or on a spread sheet, I suggest you keep those columns.

If we now examine this a bit closer, we see that we could write that \([\text{beats/Rd}]\) is:

\[
[\text{beats/Rd}] = \frac{(2\times30) \times 23 \times 72}{6 \times 23}
\]

You should note that in the units equation (rows 3 & 4) the Rf in columns 1 and 2 cancel as do the Re in columns 2 and 3. The result is \([\text{beats/Rd}]\). It can also be seen that the 23 in the numerator (i.e. row 1) will cancel with the denominator (i.e. row 2). One might expect that since Gear E is an idler gear. It has nothing to do with the time equation at all and we could have left it out. We’ve included it so that we fully understand the train.
Let’s now break this chain of thought and skip to the other end of the gear train with the same type of analysis. We might expect the last term in our tabular equation to look something like this. We know that Arbor A turns one revolution per one hour. This looks good since we want to end up with [beats/hour] and this looks about right.

Now we should be able to fill in the entire table for the gear ratio equation. It will be:

<table>
<thead>
<tr>
<th>2x30</th>
<th>23</th>
<th>72</th>
<th>78</th>
<th>20</th>
<th>32</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>23</td>
<td>7</td>
<td>78</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>beat</td>
<td>Rf</td>
<td>Re</td>
<td>Rd</td>
<td>Rc</td>
<td>Rb</td>
<td>Ra</td>
</tr>
<tr>
<td>Rf</td>
<td>Re</td>
<td>Rd</td>
<td>Rc</td>
<td>Rb</td>
<td>Ra</td>
<td>hour</td>
</tr>
</tbody>
</table>

This is a wonderful table.  
All of the Rx units terms cancel out.

Arbors B, C and E are all idlers and thus their values cancel out as well. Only the bold terms remain. The solution in [beats/hour] is:

\[
\text{[beats/hour]} = \frac{(2 \times 30 \times 72 \times 32)}{(6 \times 7)} = 3291
\]

Are we done?  Can we calculate the pendulum length?  
Let’s take a look.
Calculating the Pendulum Length

Equation 6 was the equation for the length of the pendulum.

\[ L = t^2 \left( \frac{g}{p^2} \right) \]

or in units

\[ [ft] = [sec/beat]^2 \left( \frac{ft/sec^2}{1/[beat]^2} \right) \]

or

\[ [ft] = \left( \frac{1}{[beat/sec]^2} \right) \left( \frac{ft/sec^2}{1/[beat]^2} \right) \]

Houston - “We have a problem”.

We know all these values at this point but our units are messed up.

We know the reciprocal of \( t \) in terms of \([beats/hour]\) and we need it in \([beats/sec]\). Similarly, we have \( g \) expressed in \([ft]\) and \([sec]\). We really would like it in terms of \([inches]\) and \([hours]\).

We must convert all these numbers so that our units are consistent and in the terms we want - before we make any calculations.

Perhaps there is an easier way.

The Magic Number Table

Let’s look at the problem of consistent units.

Converting beats per hour to beats per second should be easy. However, let’s see if we can’t establish a generic solution for a number of options. It should work for \([beats/unit of time]\) so we can look up the constant for sec, min or hours. Similarly it would be nice to have \( L \) expressed in either feet or inches.

The key to all of this is the constant \( (g/\pi^2) \). We can get out of this dilemma by calculating the constant for a number of conditions. Let’s take the easy one first letting \( g \) be 32.2 \([ft/sec^2]\) and \( \pi \) be 3.1416 \([1/beat]\). Our tabular equation shows:

<table>
<thead>
<tr>
<th>g</th>
<th>32.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^2 )</td>
<td></td>
</tr>
<tr>
<td>ft/sec²</td>
<td>ft</td>
</tr>
<tr>
<td>(1/beat)²</td>
<td>sec²</td>
</tr>
</tbody>
</table>

Thus if we had our train expressed in \([beats/sec]\) we could divide 3.265 \([ft (beats/sec)^2]\) by the \([beats/sec]\) squared and we would have the length of the pendulum \([ft]\).
We can convert this constant to inches by the usual 12 [in/ft] conversion.

<table>
<thead>
<tr>
<th>ft beats²</th>
<th>in</th>
<th>in beats²</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.265</td>
<td>12</td>
<td>39.15</td>
</tr>
</tbody>
</table>

We will make one more conversion and then let the reader work out the rest.

The conversion of \((\text{beats/sec})^2\) to \((\text{beats/min})^2\) requires that the terms be multiplied by 60 [sec/min] twice as follows:

Thus we would divide 11745 [ft (beats/min)²] by the square of our train number [beats/min] and would have the pendulum length in feet.

We can summarize this type of calculation with the values in the following table. It provides constants for three [beat/unit time] sec, min, & hours, and two [linear units] inches and feet. It should be a useful reference for the reader.
This table provides the generic solution to many trains expressed in English units. It is based on a value of \( g \) of 32.2 \( \text{ft/sec}^2 \). Users should recalculate the table based on the actual value of \( g \) for their location. The same sort of table could be created for metric units.

### Calculating the Pendulum Length

The pendulum length is calculated by dividing the table constant by the square of the \([\text{beats/unit of time}]\) value.

Let’s use a table value to determine the pendulum length for our original gear train. Our analysis of the gear train produced the number 3291 \([\text{beats/hour}]\).

We can now easily calculate the pendulum length [in]:

\[
\begin{array}{ccc}
507391407 & 1 & 46.84 \\
(3291)^2 & \text{in} & \\
507391407 & (\text{beats/hour})^2 & \text{in}
\end{array}
\]

At long last, we find that our pendulum will be 46.84 inches long - that’s to the center of gravity. In real life, trial and error would take over from here and the owner would have to adjust a real pendulum from here until the length was just right.

<table>
<thead>
<tr>
<th>Calc Units of The Train</th>
<th>Desired Units of L</th>
<th>Constant</th>
<th>Units of the Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>beats/sec</td>
<td>ft</td>
<td>3.2625</td>
<td>ft ((\text{beats/sec})^2)</td>
</tr>
<tr>
<td>beats/sec</td>
<td>in</td>
<td>39.1509</td>
<td>in ((\text{beats/sec})^2)</td>
</tr>
<tr>
<td>beats/ min</td>
<td>ft</td>
<td>11745</td>
<td>ft ((\text{beats/ min})^2)</td>
</tr>
<tr>
<td>beats/ min</td>
<td>in</td>
<td>140942</td>
<td>in ((\text{beats/ min})^2)</td>
</tr>
<tr>
<td>beats/ hour</td>
<td>ft</td>
<td>42282617</td>
<td>ft ((\text{beats/ hour})^2)</td>
</tr>
<tr>
<td>beats/ hour</td>
<td>in</td>
<td>507391407</td>
<td>in ((\text{beats/ hour})^2)</td>
</tr>
</tbody>
</table>
Summary

In this article we:

• made up a gear train which had three idler gears in it. We carried them along in the analysis until they fell out naturally just from the process. We did not make prior judgement as to what stayed in and what didn’t stay in.

• utilized a tabular equation format which forced us into a discipline of carrying along the units of the variables so we did not end up with something which did not make sense

• calculated a table of generic constants which will enable the reader to calculate the length of a pendulum once the [beats/unit of time] is known when the equations are in English units. This is a good reference table for the reader.

• calculated the length of the pendulum for the train I made up.

It is my hope that this article will aid others in understanding the basic concepts behind calculating the length of a pendulum based upon a wheel count.

My thanks to Bill Bugert for taking the time to struggle through the analysis of the units of the equations with me in the original article. Thanks also to David LaBounty who pointed out a lot of my typos in this version. I appreciate their help.